

# ESTIMATES OF GROMOV'S BOX DISTANCE

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**ABSTRACT.** In 1999, M. Gromov introduced the box distance function  $\square_\lambda$  on the space of all mm-spaces. In this paper, by using the method of T. H. Colding (cf. [1, Lemma 5.10]), we estimate  $\square_\lambda(\mathbb{S}^n, \mathbb{S}^m)$  and  $\square_\lambda(\mathbb{C}P^n, \mathbb{C}P^m)$ , where  $\mathbb{S}^n$  is the  $n$ -dimensional unit sphere in  $\mathbb{R}^{n+1}$  and  $\mathbb{C}P^n$  is the  $n$ -dimensional complex projective space equipped with the Fubini-Study metric. In particular, we give the complete answer to an Exercise of Gromov's Green book (cf. [4, Section 3 $\frac{1}{2}$ .18]). We also estimate  $\square_\lambda(SO(n), SO(m))$  from below, where  $SO(n)$  is the special orthogonal group.

## 1. INTRODUCTION

In 1999, M. Gromov developed the theory of mm-spaces in [4, Chapter 3 $\frac{1}{2}$ +] by introducing two distance functions, called the *box distance function*  $\square_\lambda$  and the *observable distance function*  $\underline{H}_\lambda \mathcal{L}_1$ , on the space  $\mathcal{X}$  of all isomorphic class of mm-spaces. Here, an *mm-space* is a triple  $(X, d_X, \mu_X)$ , where  $d_X$  is a complete separable metric on a set  $X$  and  $\mu_X$  a finite Borel measure on  $(X, d_X)$ . The notion of the distance function  $\square_\lambda$  is considered as a natural extension of the Gromov-Hausdorff distance function to the space  $\mathcal{X}$ . On the other hand, the notion of the distance function  $\underline{H}_\lambda \mathcal{L}_1$  is related to measure concentration. Roughly speaking, “measure concentration” amounts to saying that the push-forward measures  $f_{n*}(\mu_n)$  on  $\mathbb{R}$  concentrate to a point for any sequence of 1-Lipschitz functions  $f_n : (X_n, d_n, \mu_n) \rightarrow \mathbb{R}$ . For instance, the unit spheres in Euclidean spaces  $\{\mathbb{S}^n\}_{n=1}^\infty$ , the complex projective spaces  $\{\mathbb{C}P^n\}_{n=1}^\infty$  equipped with the Fubini-Study metrics, and the special orthogonal groups  $\{SO(n)\}_{n=1}^\infty$  have that property. He defined the distance  $\underline{H}_\lambda \mathcal{L}_1(X, Y)$  by using the Hausdorff distance between the space of 1-Lipschitz functions on  $X$  and that on  $Y$ , and showed that a sequence  $\{X_n\}_{n=1}^\infty$  of mm-spaces concentrates if and only if the sequence  $\{X_n\}_{n=1}^\infty$  converges to a one-point space with respect to the distance function  $\underline{H}_\lambda \mathcal{L}_1$ .

The topology on  $\mathcal{X}$  determined by  $\square_\lambda$  is strictly stronger than that of  $\underline{H}_\lambda \mathcal{L}_1$ . In fact, the sequences  $\{\mathbb{S}^n\}_{n=1}^\infty$ ,  $\{\mathbb{C}P^n\}_{n=1}^\infty$ , and  $\{SO(n)\}_{n=1}^\infty$  are all divergent with respect to the distance  $\square_\lambda$  (see Proposition 3.1). This is related to the following exercise in Gromov's book:

**Exercise** (cf. [4, Section 3 $\frac{1}{2}$ .18]). *Estimate the distance  $\square_\lambda(\mathbb{S}^n, \mathbb{S}^m)$ .*

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To solve the exercise, applying a method of [1, Lemma 5.10], we will estimate  $\square_\lambda(M, N)$  from below for compact Riemannian manifolds  $M$  and  $N$  with positive Ricci curvatures and the volume measures satisfying a homogeneity condition (see Lemma 3.4). As a result, we get the following proposition:

**Proposition 1.1.** *Assume that two sequences  $\{n_k\}_{k=1}^\infty, \{m_k\}_{k=1}^\infty$  of natural numbers satisfy  $n_k \leq C_1 k, m_k \leq C_2 k$  and  $|n_k - m_k| \geq C_3 k, k = 1, 2, \dots$ , for some positive constants  $C_1, C_2, C_3$ . Then, we have*

$$\liminf_{k \rightarrow \infty} \square_1(\mathbb{S}^{n_k}, \mathbb{S}^{m_k}), \liminf_{k \rightarrow \infty} \square_1(\mathbb{C}P^{n_k}, \mathbb{C}P^{m_k}) \geq \min \left\{ 2^{-\frac{C_1}{C_3}} \pi^{-\frac{C_2}{C_3}}, 2^{-\frac{C_2}{C_3}} \pi^{-\frac{C_1}{C_3}} \right\}.$$

In particular, if in addition  $|n_k - m_k| \geq C_4 k^\alpha, k = 1, 2, \dots$ , holds for some constant  $C_4 > 0$  and a number  $\alpha > 1$ , then we have

$$\lim_{k \rightarrow \infty} \square_1(\mathbb{S}^{n_k}, \mathbb{S}^{m_k}), \lim_{k \rightarrow \infty} \square_1(\mathbb{C}P^{n_k}, \mathbb{C}P^{m_k}) = 1.$$

We estimate  $\square_\lambda(SO(n), SO(m))$  from below by the difference of their diameters (see Lemma 3.8). Consequently, we obtain the following proposition:

**Proposition 1.2.** *Assume that two sequences  $\{n_k\}_{k=1}^\infty, \{m_k\}_{k=1}^\infty$  of natural numbers satisfy  $n_k \leq C_1 k, m_k \leq C_2 k$  and  $|n_k - m_k| \geq C_3 \sqrt{k}, k = 1, 2, \dots$ , for some positive constants  $C_1, C_2, C_3$ . Then, we have*

$$\liminf_{k \rightarrow \infty} \square_1(SO(n_k), SO(m_k)) \geq \min \left\{ \frac{1}{2}, \frac{C_3}{\sqrt{C_1} + \sqrt{C_2}} \right\}.$$

In particular, if in addition  $|n_k - m_k| \geq C_4 k^\alpha, k = 1, 2, \dots$ , holds for some constant  $C_4 > 0$  and a number  $\alpha > 1/2$ , then we have

$$\liminf_{k \rightarrow \infty} \square_1(SO(n_k), SO(m_k)) \geq \frac{1}{2}.$$

As is related to the above Gromov's exercise, we also prove the following proposition. This proposition is also mentioned by Gromov in [4, Section 3 $\frac{1}{2}$ .3 Exercise (e)].

**Proposition 1.3.** *We have*

$$\square_1(\mathbb{S}^n, \mathbb{S}^{n-1}), \square_1(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \rightarrow 0$$

as  $n \rightarrow \infty$ .

## 2. PRELIMINARIES

### 2.1. Definition of Gromov's box distance function $\square_\lambda$ .

**Definition 2.1.** Let  $\lambda \geq 0$  and  $(X, \mu)$  be a measure space with  $\mu(X) < +\infty$ . For two maps  $d_1, d_2 : X \times X \rightarrow \mathbb{R}$ , we define a number  $\square_\lambda(d_1, d_2)$  as the infimum of  $\varepsilon > 0$  such that there exists a measurable subset  $T_\varepsilon \subseteq X$  of measure at least  $\mu(X) - \lambda\varepsilon$  satisfying  $|d_1(x, y) - d_2(x, y)| \leq \varepsilon$  for any  $x, y \in T_\varepsilon$ .

It is easy to see that this is a distance function on the set of all functions on  $X \times X$ , and the two distance functions  $\square_\lambda$  and  $\square_{\lambda'}$  are equivalent to each other for any  $\lambda, \lambda' > 0$ .

**Definition 2.2** (parameter). Let  $X$  be an mm-space and  $\mu(X) = m$ . Then, there exists a Borel measurable map  $\varphi : [0, m] \rightarrow X$  with  $\varphi_*(\mathcal{L}) = \mu$ , where  $\mathcal{L}$  stands for the Lebesgue measure on  $[0, m]$ . We call  $\varphi$  a *parameter* of  $X$ .

Note that if the support of  $X$  is not a one-point, then its parameter is not unique.

**Definition 2.3** (Gromov's box distance function). If two mm-spaces  $X, Y$  satisfy  $\mu_X(X) = \mu_Y(Y) = m$ , we define

$$\square_\lambda(X, Y) := \inf \square_\lambda(\varphi_X^* d_X, \varphi_Y^* d_Y),$$

where the infimum is taken over all parameters  $\varphi_X : [0, m] \rightarrow X$ ,  $\varphi_Y : [0, m] \rightarrow Y$ , and  $\varphi_X^* d_X$  is defined by  $\varphi_X^* d_X(s, t) := d_X(\varphi_X(s), \varphi_X(t))$  for  $s, t \in [0, m]$ . If  $\mu_X(X) < \mu_Y(Y)$ , putting  $m := \mu_X(X)$ ,  $m' := \mu_Y(Y)$ , we define

$$\square_\lambda(X, Y) := \square_\lambda\left(X, \frac{m}{m'}Y\right) + m' - m,$$

where  $(m/m')Y := (Y, d_Y, (m/m')\mu_Y)$ .

We recall that two mm-spaces are *isomorphic* to each other if there is a measure preserving isometry between the supports of their measures.  $\square_\lambda$  is a distance function on  $\mathcal{X}$  for any  $\lambda \geq 0$ . See [2, Section 1, 3] for a complete proof of that. Note that the distance functions  $\square_\lambda$  and  $\square_{\lambda'}$  are equivalent to each other for distinct  $\lambda, \lambda' > 0$ .

**2.2. Definition of observable distance functions  $H_\lambda \mathcal{L}_{t_1}$ .** For a measure space  $(X, \mu)$  with  $\mu(X) < +\infty$ , we denote by  $\mathcal{F}(X, \mathbb{R})$  the space of all functions on  $X$ . Given  $\lambda \geq 0$  and  $f, g \in \mathcal{F}(X, \mathbb{R})$ , we put

$$\text{me}_\lambda(f, g) := \inf\{\varepsilon > 0 \mid \mu(\{x \in X \mid |f(x) - g(x)| \geq \varepsilon\}) \leq \lambda\varepsilon\}.$$

Note that this  $\text{me}_\lambda$  is a distance function on  $\mathcal{F}(X, \mathbb{R})$  for any  $\lambda \geq 0$  and its topology on  $\mathcal{F}(X, \mathbb{R})$  coincides with the topology of the convergence in measure for any  $\lambda > 0$ . Also, the distance functions  $\text{me}_\lambda$  for all  $\lambda > 0$  are mutually equivalent.

We recall that the *Hausdorff distance* between two closed subsets  $A$  and  $B$  in a metric space  $X$  is defined by

$$d_H(A, B) := \inf\{\varepsilon > 0 \mid A \subseteq B_\varepsilon, B \subseteq A_\varepsilon\},$$

where  $A_\varepsilon$  is a closed  $\varepsilon$ -neighborhood of  $A$ .

Let  $(X, \mu)$  be a measure space with  $\mu(X) < +\infty$ . For a semi-distance  $d$  on  $X$ , we indicate by  $\mathcal{Lip}_1(d)$  the space of all 1-Lipschitz functions on  $X$  with respect to  $d$ . Note that  $\mathcal{Lip}_1(d)$  is a closed subset in  $(\mathcal{F}(X, \mathbb{R}), \text{me}_\lambda)$  for any  $\lambda \geq 0$ .

**Definition 2.4.** For  $\lambda \geq 0$  and two semi-distance functions  $d, d'$  on  $X$ , we define

$$H_\lambda \mathcal{L}_{t_1}(d, d') := d_H(\mathcal{Lip}_1(d), \mathcal{Lip}_1(d')),$$

where  $d_H$  stands for the Hausdorff distance function in  $(\mathcal{F}(X, \mathbb{R}), \text{me}_\lambda)$ .

This  $H_\lambda \mathcal{L}_{\iota_1}$  is actually a distance function on the space of all semi-distance functions on  $X$  for all  $\lambda \geq 0$ , and the two distance functions  $H_\lambda \mathcal{L}_{\iota_1}$  and  $H_{\lambda'} \mathcal{L}_{\iota_1}$  are equivalent to each other for any  $\lambda, \lambda' > 0$ .

**Lemma 2.5.** *For any two semi-distance functions  $d, d'$  on  $X$ , we have*

$$H_\lambda \mathcal{L}_{\iota_1}(d, d') \leq \square_\lambda(d, d').$$

*Proof.* For any  $\varepsilon > 0$  with  $\square_\lambda(X, Y) < \varepsilon$ , there exists a measurable subset  $T_\varepsilon \subseteq X$  such that  $\mu(X \setminus T_\varepsilon) \leq \lambda\varepsilon$  and  $|d(x, y) - d'(x, y)| \leq \varepsilon$  for any  $x, y \in T_\varepsilon$ . Given arbitrary  $f \in \mathcal{Lip}_1(d)$ , we define  $\tilde{f} \in \mathcal{F}(X, \mathbb{R})$  by  $\tilde{f}(x) := \inf\{f(y) + d'(x, y) \mid y \in T_\varepsilon\}$ . We see easily that  $\tilde{f} \in \mathcal{Lip}_1(d')$  and  $\tilde{f}(x) \leq f(x)$  for any  $x \in T_\varepsilon$ . Taking any  $x \in T_\varepsilon$ , we have

$$\begin{aligned} |f(x) - \tilde{f}(x)| &= f(x) - \tilde{f}(x) \\ &= \sup\{f(x) - f(y) - d'(x, y) \mid y \in T_\varepsilon\} \\ &\leq \sup\{d(x, y) - d'(x, y) \mid y \in T_\varepsilon\} \\ &\leq \varepsilon. \end{aligned}$$

Therefore, we get  $\text{me}_\lambda(f, \tilde{f}) \leq \varepsilon$ , which implies  $\mathcal{Lip}_1(d) \subseteq (\mathcal{Lip}_1(d'))_\varepsilon$ . Similarly, we also have  $\mathcal{Lip}_1(d') \subseteq (\mathcal{Lip}_1(d))_\varepsilon$ , which yields  $H_\lambda \mathcal{L}_{\iota_1}(d, d') \leq \varepsilon$ . This completes the proof.  $\square$

**Definition 2.6** (Observable distance function). *If two mm-spaces  $X, Y$  satisfy  $\mu_X(X) = \mu_Y(Y) = m$ , we define*

$$\underline{H}_\lambda \mathcal{L}_{\iota_1}(X, Y) := \inf H_\lambda \mathcal{L}_{\iota_1}(\varphi_X^* d_X, \varphi_Y^* d_Y),$$

where the infimum is taken over all parameters  $\varphi_X : [0, m] \rightarrow X$ ,  $\varphi_Y : [0, m] \rightarrow Y$ . If  $\mu_X(X) < \mu_Y(Y)$ , putting  $m := \mu_X(X)$ ,  $m' := \mu_Y(Y)$ , we define

$$\underline{H}_\lambda \mathcal{L}_{\iota_1}(X, Y) := \underline{H}_\lambda \mathcal{L}_{\iota_1}\left(X, \frac{m}{m'} Y\right) + m' - m.$$

$\underline{H}_\lambda \mathcal{L}_{\iota_1}$  is a distance function on  $\mathcal{X}$  for any  $\lambda \geq 0$ . See [2, Section 3] for a complete proof of that. Note that the distance functions  $\underline{H}_\lambda \mathcal{L}_{\iota_1}$  and  $\underline{H}_{\lambda'} \mathcal{L}_{\iota_1}$  are equivalent to each other for any  $\lambda, \lambda' > 0$ .

For a Borel measure  $\nu$  on  $\mathbb{R}$  with  $m := \nu(\mathbb{R}) < +\infty$  and  $\kappa > 0$ , we put

$$\text{diam}(\nu, m - \kappa) := \inf\{\text{diam } Y \mid Y \subseteq \mathbb{R} \text{ is a Borel subset such that } \nu_Y(Y) \geq m - \kappa\},$$

and call it the *partial diameter* on  $\nu$ .

**Definition 2.7** (Observable diameter). Let  $(X, d, \mu)$  be an mm-space and let  $m := \mu(X)$ . For any  $\kappa > 0$  we define the *observable diameter* of  $X$  by

$$\text{diam}(X \xrightarrow{\text{Lip}_1} \mathbb{R}, m - \kappa) := \sup\{\text{diam}(f_*(\mu), m - \kappa) \mid f : X \rightarrow \mathbb{R} \text{ is an 1-Lipschitz function}\}.$$

The idea of the observable diameter came from the quantum and statistical mechanics, that is, we think of  $\mu$  as a state on a configuration space  $X$  and  $f$  is interpreted as an observable. We define a sequence  $\{X_n\}_{n=1}^\infty$  of mm-spaces is a *Lévy family* if

$\text{diam}(X_n \xrightarrow{\text{Lip}_1} \mathbb{R}, m_n - \kappa) \rightarrow 0$  as  $n \rightarrow \infty$  for any  $\kappa > 0$ , where  $m_n$  is the total measure of the mm-space  $X_n$ . This is equivalent to that for any  $\varepsilon > 0$  and any sequence  $\{f_n : X_n \rightarrow \mathbb{R}\}_{n=1}^\infty$  of 1-Lipschitz functions, we have

$$(\diamond) \quad \mu_n(\{x \in X_n \mid |f_n(x) - m_{f_n}| \geq \varepsilon\}) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

where  $m_{f_n}$  is a some constant determined by  $f_n$ .

**Example 2.8.** Let  $\{M_n\}_{n=1}^\infty$  be a sequence of compact connected Riemannian manifolds. Let  $d_n$  be the Riemannian distance on  $M_n$  and  $\mu_n$  be its Riemannian volume measure normalized as  $\mu_n(M_n) = 1$ . Assume that  $\text{Ric}_{M_n} \geq \kappa_n \rightarrow +\infty$  as  $n \rightarrow \infty$ . Then, by virtue of Lévy-Gromov's isoperimetric inequality, the sequence  $\{M_n\}_{n=1}^\infty$  is a Lévy family (cf. [5, Section 1, Remark 2]). For example,  $\{\mathbb{S}^n\}_{n=1}^\infty$  and  $\{\mathbb{C}P^n\}_{n=1}^\infty$  are Lévy families. Recall that the Fubini-Study metric on  $\mathbb{C}P^n$  is the unique Riemannian metric on  $\mathbb{C}P^n$  such that the canonical projection  $\mathbb{S}^{2n+1} \rightarrow \mathbb{C}P^n$  is a Riemannian submersion. Since  $\text{Ric}_{SO(n)} \geq (n-1)/4$ , the sequence  $\{SO(n)\}_{n=1}^\infty$  is Lévy family. Since the distance function induced from the Hilbert-Schmidt norm on  $SO(n)$  is not greater than that of the Riemannian distance function,  $\{SO(n)\}_{n=1}^\infty$  is Lévy family with respect to also the Hilbert-Schmidt norms.

**Example 2.9** (Hamming cube). Let  $\mu_n$  be the normalized counting measure on  $\{0, 1\}^n$  and  $d_n$  be the *Hamming distance function* on  $\{0, 1\}^n$ , that is,

$$d_n((x_i)_{i=1}^n, (y_i)_{i=1}^n) := \frac{1}{n} \text{Card}(\{i \in \{1, \dots, n\} \mid x_i \neq y_i\}).$$

The mm-space  $\{0, 1\}^n$  is called the *Hamming cube*. The sequence  $\{\{0, 1\}^n\}_{n=1}^\infty$  is a Lévy family (cf. [4, Section 3 $\frac{1}{2}$ .42]).

Gromov showed the following proposition by considering a constant  $m_{f_n}$  in  $(\diamond)$  as a Lipschitz function from a one-point space  $\{*_n\}$  with total measure  $\mu_n(X_n)$ .

**Proposition 2.10.** [4, Section 3 $\frac{1}{2}$ .45] *A sequence  $\{X_n\}_{n=1}^\infty$  of mm-spaces is a Lévy family if and only if  $\underline{H}_\lambda \mathcal{L}_{t_1}(X_n, \{*_n\}) \rightarrow 0$  as  $n \rightarrow \infty$  for any  $\lambda > 0$ .*

### 3. ESTIMATES OF GROMOV'S BOX DISTANCE FUNCTION

Let  $X$  be a metric space. Denote by  $B_X(x, r)$  the closed ball in  $X$  centered at  $x \in X$  with radius  $r > 0$ . A Borel measure  $\mu$  on  $X$  is said to be *uniformly distributed* if

$$0 < \mu(B_X(x, r)) = \mu(B_X(y, r)) < +\infty$$

for any  $r > 0$  and  $x, y \in X$ .

From Lemma 2.5, we see that the topology on  $\mathcal{X}$  determined by  $\underline{\square}_\lambda$  is not weaker than that of  $\underline{H}_\lambda \mathcal{L}_{t_1}$  for any  $\lambda \geq 0$ . For a Borel measure  $\mu$  on a metric space, we denote by  $\text{Supp } \mu$  its support.

**Proposition 3.1.** *Let  $\{(X_n, d_n, \mu_n)\}_{n=1}^\infty$  be a Lévy family such that  $\mu_n$  is uniformly distributed Borel probability measure satisfying  $X_n = \text{Supp } \mu_n$  and  $\inf_{n \in \mathbb{N}} \text{diam } X_n > 0$ . Then, the sequence  $\{X_n\}_{n=1}^\infty$  does not converge with respect to the distance function  $\square_\lambda$  for any  $\lambda \geq 0$ .*

*Proof.* Suppose that  $\{X_n\}_{n=1}^\infty$  converges and let  $X$  be its limit. Since  $\{X_n\}_{n=1}^\infty$  is a Lévy family, by using Proposition 2.10,  $X$  must be a one-point space. Fix  $\varepsilon > 0$  with  $\varepsilon < \min\{3, \inf_{n \in \mathbb{N}} \text{diam } X_n\}/3$ . For any sufficiently large  $n \in \mathbb{N}$ , there exists a parameter  $\varphi_n : [0, 1] \rightarrow X_n$  of  $X_n$  and Borel subset  $T_n \subseteq [0, 1]$  such that  $\mathcal{L}(T_n) > 1 - \varepsilon/2$  and  $d_n(\varphi_n(s), \varphi_n(t)) < \varepsilon/2$  for any  $s, t \in T_n$ . Fix a point  $t_n \in T_n$ . There exists a point  $x_n \in X_n$  such that  $d_n(\varphi_n(t_n), x_n) \geq \text{diam } X_n/3 > \varepsilon$  and hence  $B_{X_n}(\varphi_n(t_n), \varepsilon/2) \cap B_{X_n}(x_n, \varepsilon/2) = \emptyset$ . Therefore, we get

$$\begin{aligned} 1 &\geq \mu_n(B_{X_n}(\varphi_n(t_n), \varepsilon/2) \cup B_{X_n}(x_n, \varepsilon/2)) \\ &= 2\mu_n(B_{X_n}(\varphi_n(t_n), \varepsilon/2)) \\ &= 2\mathcal{L}(\varphi_n^{-1}(B_{X_n}(\varphi_n(t_n), \varepsilon/2))) \geq 2\mathcal{L}(T_n) \geq 2 - \varepsilon > 1, \end{aligned}$$

which gives a contradiction. This completes the proof.  $\square$

From Proposition 3.1, we see that many Lévy families such as  $\{\mathbb{S}^n\}_{n=1}^\infty$ ,  $\{\mathbb{C}P^n\}_{n=1}^\infty$ ,  $\{SO(n)\}_{n=1}^\infty$ , and  $\{\{0, 1\}^n\}_{n=1}^\infty$  have no convergent subsequences with respect to the distance function  $\square_\lambda$ . Therefore, the distance function  $\square_\lambda$  determines the topology on  $\mathcal{X}$  strictly stronger than that of the distance function  $\underline{H}_\lambda \mathcal{L}_{\iota_1}$  for any  $\lambda > 0$ . However, since the proof of Proposition 3.1 is by contradiction, we do not estimate  $\square_\lambda(X_n, X_m)$  from below for  $n, m \in \mathbb{N}$ .

The proof of the following proposition is an analogue of the proof of [1, Lemma 5.10]

**Lemma 3.2.** *Let  $(X, d_X, \mu_X), (Y, d_Y, \mu_Y)$  be mm-spaces and assume that  $\mu_X, \mu_Y$  are uniformly distributed Borel probability measures. Denote by  $v_X(r)$  (respectively,  $v_Y(r)$ ) the measure of a closed ball of  $X$  (respectively,  $Y$ ) with radius  $r > 0$  and assume that  $v_X(a+c) \leq (1-c)v_Y(a/2)$  for some  $a, c > 0$  with  $c < 1$ . Then, we have  $\square_1(X, Y) \geq c$ .*

*Proof.* Let us prove the lemma by contradiction. Suppose that  $\square_1(X, Y) < c$ . Then, there exist compact subset  $T \subseteq [0, 1]$  and two parameters  $\varphi_X : [0, 1] \rightarrow X$ ,  $\varphi_Y : [0, 1] \rightarrow Y$  such that

- (1)  $\mathcal{L}(T) > 1 - c$ ,
- (2)  $\varphi_X|_T : T \rightarrow X$ ,  $\varphi_Y|_T : T \rightarrow Y$  are continuous,
- (3)  $|d_X(\varphi_X(s), \varphi_X(t)) - d_Y(\varphi_Y(s), \varphi_Y(t))| < c$  for any  $s, t \in T$ .

By (1) and (2),  $\varphi_Y(T)$  is compact. Put

$$\begin{aligned} l &:= \max\{k \in \mathbb{N} \mid \text{there exist points } p_i, i = 1, \dots, k, \text{ such that} \\ &\quad B_Y(p_i, a/2) \cap B_Y(p_j, a/2) = \emptyset \text{ for any } i, j \text{ with } i \neq j\}. \end{aligned}$$

Then, there exist points  $p_i$ ,  $i = 1, \dots, k$ , such that  $B_Y(p_i, a/2) \cap B_Y(p_j, a/2) = \emptyset$  for any  $i, j$  with  $i \neq j$ . Hence, we get

$$1 \geq \mu_Y\left(\bigcup_{i=1}^l B_Y(p_i, a/2)\right) = \sum_{i=1}^l \mu_Y(B_Y(p_i, a/2)) = l \cdot v_Y(a/2).$$

It also follows from the definition of  $l$  that  $\varphi_Y(T) \subseteq \bigcup_{i=1}^l B_Y(p_i, a)$ . For any  $i = 1, \dots, l$ , we fix  $t_i \in T$  with  $p_i = \varphi_Y(t_i)$ .

**Claim 3.3.**

$$\varphi_X(T) \subseteq \bigcup_{i=1}^l B_X(\varphi_X(t_i), a + c).$$

*Proof.* Take an arbitrary  $q = \varphi_X(s) \in \varphi_X(T)$ ,  $s \in T$ . Since  $\varphi_Y(s) \in \varphi_Y(T) \subseteq \bigcup_{i=1}^l B_Y(p_i, a)$ , there exists  $i$  with  $1 \leq i \leq l$  such that  $d_Y(\varphi_Y(s), p_i) \leq a$ . Therefore, by using (2), we obtain

$$d_X(\varphi_X(s), \varphi_X(t_i)) < d_Y(\varphi_Y(s), p_i) + c \leq a + c.$$

This completes the proof of the claim.  $\square$

Applying Claim 3.3, we get

$$1 \leq \sum_{i=1}^l \frac{\mu_X(B_X(\varphi_X(t_i), a + c))}{\mu_X(\varphi_X(T))} = l \cdot \frac{v_X(a + c)}{\mu_X(\varphi_X(T))} \leq \frac{v_X(a + c)}{v_Y(a/2) \cdot \mu_X(\varphi_X(T))}.$$

Since  $\mu_X(\varphi_X(T)) \geq \mathcal{L}(\varphi_X^{-1}(\varphi_X(T))) \geq \mathcal{L}(T) > 1 - c$ , we obtain

$$1 \leq \frac{v_X(a + c)}{v_Y(a/2) \cdot \mu_X(\varphi_X(T))} < \frac{v_X(a + c)}{v_Y(a/2) \cdot (1 - c)} \leq 1,$$

which is a contradiction. This completes the proof of Lemma 3.2.  $\square$

For a compact Riemannian manifold  $M$ , we denote by  $\text{vol}(M)$  the total Riemannian volume of  $M$ . We indicate by  $\Gamma$  the Gamma function.

**Lemma 3.4.** *Let  $M$  (respectively,  $N$ ) be an  $m$  (respectively,  $n$ )-dimensional compact Riemannian manifold having the uniformly distributed Riemannian volume measure. Assume that  $\text{Ric}_M \geq (m - 1)\kappa_1 > 0$  and  $\text{Ric}_N > 0$ , and put  $a_N := \text{vol}(N)/\text{vol}(\mathbb{S}^n)$ . If a positive number  $c$  with  $c < 1$  satisfies*

$$c^{n-m} \leq (1 - c) \frac{na_N(\kappa_1)^{m/2} \Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n}{2}\right)}{m2^{n+1} \pi^{m-1} \Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n+1}{2}\right)} \text{ and } c\sqrt{\kappa_1} \leq \pi,$$

*then we have  $\square_1(M, N) \geq c$ .*

*Proof.* From the Bishop-Gromov volume comparison theorem, we get

$$v_M(c/2) \geq v_{\mathbb{S}^m}((c\sqrt{\kappa_1})/2) = \frac{\text{vol}(\mathbb{S}^{m-1})}{\text{vol}(\mathbb{S}^m)} \int_0^{(c\sqrt{\kappa_1})/2} \sin^{m-1} \theta d\theta.$$

From  $c\sqrt{\kappa_1} \leq \pi$ , we have  $\sin \theta \geq (\pi\theta)/2$  for any  $\theta \in [0, (c\sqrt{\kappa_1})/2]$ . Hence, we obtain

$$v_M(c/2) \geq \frac{2^{m-1} \text{vol}(\mathbb{S}^{m-1})}{\pi^{m-1} \text{vol}(\mathbb{S}^m)} \int_0^{(c\sqrt{\kappa_1})/2} \theta^{m-1} d\theta = \frac{c^m (\kappa_1)^{m/2} \text{vol}(\mathbb{S}^{m-1})}{2m\pi^{m-1} \text{vol}(\mathbb{S}^m)}$$

Let  $\kappa_2$  be a positive number such that  $\text{Ric}_N \geq (n-1)\kappa_2$ . We also obtain from the Bishop inequality that

$$v_N(2c) \leq \frac{v_{\mathbb{S}^n}(2c\sqrt{\kappa_2})}{a_N(\kappa_2)^{n/2}} = \frac{\text{vol}(\mathbb{S}^{n-1})}{a_N(\kappa_2)^{n/2} \text{vol}(\mathbb{S}^n)} \int_0^{2c\sqrt{\kappa_2}} \sin^{n-1} \theta d\theta < \frac{(2c)^n \text{vol}(\mathbb{S}^{n-1})}{na_N \text{vol}(\mathbb{S}^n)}.$$

Recall that  $\text{vol}(\mathbb{S}^n) = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$ . Therefore, combining above calculations with Lemma 3.2, we complete the proof.  $\square$

*Proof of Proposition 1.1.* Without loss of generality, it may be assumed that  $n_k \geq m_k$ .

First, we consider the case of  $\{\mathbb{S}^n\}_{n=1}^\infty$ . From the assumption, we have  $c^{n_k-m_k} \leq c^{C_3 k}$  for any  $0 < c < 1$ . Substituting  $n := n_k$  and  $m := m_k$ , we estimate the right-hand side of the inequality of Lemma 3.4 by

$$(1-c) \frac{n_k \Gamma(\frac{m_k+1}{2}) \Gamma(\frac{n_k}{2})}{m_k 2^{n_k} \pi^{m_k-1} \Gamma(\frac{m_k}{2}) \Gamma(\frac{n_k+1}{2})} \geq (1-c) 2^{-C_1 k} \pi^{-C_2 k+1} \frac{n_k \Gamma(\frac{n_k}{2})}{m_k \Gamma(\frac{n_k+1}{2})}.$$

Therefore, if

$$c \leq \left\{ (1-c) \frac{n_k \Gamma(\frac{n_k}{2})}{m_k \Gamma(\frac{n_k+1}{2})} \right\}^{\frac{1}{C_3 k}} 2^{-\frac{C_1}{C_3} k} \pi^{-\frac{C_2}{C_3} k + \frac{1}{C_3 k}},$$

then we obtain from Lemma 3.4 that  $\square_1(\mathbb{S}^{n_k}, \mathbb{S}^{m_k}) \geq c$ . Since

$$\left\{ (1-c) \frac{n_k \Gamma(\frac{n_k}{2})}{m_k \Gamma(\frac{n_k+1}{2})} \right\}^{\frac{1}{C_3 k}} \rightarrow 1 \text{ as } k \rightarrow \infty,$$

we have completed the proof for  $\{\mathbb{S}^n\}_{n=1}^\infty$ .

Next, we consider  $\{\mathbb{C}P^n\}_{n=1}^\infty$ . It is well-known that  $\text{vol}(\mathbb{C}P^n) = \pi^n/n!$  and the sectional curvature of  $\mathbb{C}P^n$  is bounded from below by 1 (cf. [3, Section 3.D.2, 3.H.3]). Hence, we get

$$a_{\mathbb{C}P^n} = \frac{\Gamma(n + \frac{1}{2})}{2\sqrt{\pi}n!}.$$

For any  $0 < c < 1$ , we have  $c^{2n_k-2m_k} \leq c^{2C_3 k}$ . Substituting  $n := 2n_k$  and  $m := 2m_k$ , we calculate the right-hand side of the inequality of Lemma 3.4 by

$$(1-c) \frac{\Gamma(m_k + \frac{1}{2})}{2\sqrt{\pi}m_k 2^{2n_k+1} \pi^{2m_k-1} \Gamma(m_k)} \geq (1-c) \frac{1}{2\sqrt{\pi}C_2 k} \cdot 2^{-2C_1 k-1} \pi^{-2C_2 k+1}.$$



So, if

$$c \leq \left\{ (1-c) \frac{1}{2\sqrt{\pi}C_2k} \right\}^{\frac{1}{2C_3k}} 2^{-\frac{C_1}{C_3} - \frac{1}{2C_3k}} \pi^{-\frac{C_2}{C_3} + \frac{1}{2C_3k}},$$

then we get by using Lemma 3.4 that  $\square_1(\mathbb{C}P^{n_k}, \mathbb{C}P^{m_k}) \geq c$ . Since

$$\left\{ (1-c) \frac{1}{2\sqrt{\pi}C_2k} \right\}^{\frac{1}{2C_3k}} \rightarrow 1 \text{ as } k \rightarrow \infty,$$

we complete the proof of the proposition.  $\square$

Note that  $\text{diam}(\mathcal{X}_1, \square_1) = 1$ , where  $\mathcal{X}_1$  is the space of all mm-spaces with Borel probability measures.

**Lemma 3.5** (J. Christensen, c.f. [6, Section 3.3]). *Let  $X$  be a metric space and  $\mu, \nu$  are uniformly distributed Borel measures on  $X$ . Then, there exists a positive number  $c > 0$  such that  $\mu = c\nu$ .*

*Proof of Proposition 1.3.* We identify  $\mathbb{S}^{n-1}$  with  $\{(x_1, \dots, x_n, 0) \in \mathbb{S}^n \mid (x_1, \dots, x_n) \in \mathbb{S}^{n-1}\}$ . Given an arbitrary  $\varepsilon > 0$ , since the sequence  $\{\mathbb{S}^n\}_{n=1}^\infty$  is a Lévy family, we have  $r_n := \mu_n((\mathbb{S}^{n-1})_\varepsilon) \rightarrow 1$  as  $n \rightarrow \infty$ . Hence, there is  $m \in \mathbb{N}$  such that  $1 - r_n < \varepsilon$  for any  $n \geq m$ . Suppose that  $n \geq m$ . Taking two parameters  $\Phi_1 : [0, r_n] \rightarrow (\mathbb{S}^{n-1})_\varepsilon$  and  $\Phi_2 : (r_n, 1] \rightarrow \mathbb{S}^n \setminus (\mathbb{S}^{n-1})_\varepsilon$ , we define a Borel measurable map  $\Phi : [0, 1] \rightarrow \mathbb{S}^n$  by

$$\Phi(t) := \begin{cases} \Phi_1(t) & t \in [0, r_n], \\ \Phi_2(t) & t \in (r_n, 1]. \end{cases}$$

The map  $\Phi$  is a parameter of  $\mathbb{S}^n$ . Let  $\psi : \mathbb{S}^n \setminus \{(0, \dots, 0, 1), (0, \dots, 0, -1)\} \rightarrow \mathbb{S}^{n-1}$  be the projection, that is,  $\psi(x)$  is the unique element of  $\mathbb{S}^{n-1}$  satisfying  $d_n(x, \psi(x)) = d_n(x, \mathbb{S}^{n-1})$ . Put  $\varphi_1 := \psi \circ \Phi_1 : [0, r_n] \rightarrow \mathbb{S}^{n-1}$ .

**Claim 3.6.**  $\varphi_{1*}(\mathcal{L}) = r_n \mu_{n-1}$ .

*Proof.* Take any Borel subset  $A \subseteq \mathbb{S}^{n-1}$ . For any  $g \in SO(n-1)$ , we have

$$\varphi_{1*}(\mathcal{L})(gA) = r_n \mu_n(\psi^{-1}(gA)) = r_n \mu_n(\psi^{-1}(A)) = \varphi_{1*}(\mathcal{L})(A).$$

Hence,  $\varphi_{1*}(\mathcal{L})$  is a  $SO(n-1)$ -invariant Borel measure. From Lemma 3.5, we complete the proof of the claim.  $\square$

Taking a parameter  $\phi : (0, 1] \rightarrow \mathbb{S}^{n-1}$  of  $\mathbb{S}^{n-1}$ , we define a Borel measurable map  $\varphi_2 : (r_n, 1] \rightarrow \mathbb{S}^{n-1}$  by  $\varphi_2(t) := \phi((t - r_n)/(1 - r_n))$ . Then, we have  $\varphi_{2*}(\mathcal{L}) = (1 - r_n)\mathcal{L}$ . Therefore, defining a Borel measurable map  $\varphi : [0, 1] \rightarrow \mathbb{S}^{n-1}$  by

$$\varphi(t) := \begin{cases} \varphi_1(t) & t \in [0, r_n], \\ \varphi_2(t) & t \in (r_n, 1], \end{cases}$$

we see that the map  $\varphi$  is a parameter of  $\mathbb{S}^{n-1}$ . Since

$$\begin{aligned} & |d_n(\Phi(s), \Phi(t)) - d_{n-1}(\varphi(s), \varphi(t))| \\ &= |d_n(\Phi_1(s), \Phi_1(t)) - d_{n-1}(\varphi_1(s), \varphi_1(t))| \leq 2\varepsilon \end{aligned}$$

for any  $s, t \in [0, r_n]$ , we get

$$\square_1(\mathbb{S}^n, \mathbb{S}^{n-1}) \leq \square_1(\Phi^* d_n, \varphi^* d_{n-1}) \leq \max\{2\varepsilon, 1 - r_n\} = 2\varepsilon.$$

Consequently, we obtain  $\square_1(\mathbb{S}^n, \mathbb{S}^{n-1}) \rightarrow 0$  as  $n \rightarrow \infty$ . A similar argument shows that  $\square_1(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \rightarrow 0$  as  $n \rightarrow \infty$ . This completes the proof of Proposition 1.3.  $\square$

**Lemma 3.7.** *For any  $n, m \in \mathbb{N}$ , we have*

$$\square_1(SO(n), SO(m)) \geq c(n, m) := \min \left\{ \frac{1}{2}, |\text{diam } SO(n) - \text{diam } SO(m)| \right\}.$$

*Proof.* Suppose that  $n > m$  and  $\square_1(SO(n), SO(m)) < c(n, m)$ . There exist compact subset  $T \subseteq [0, 1]$  and two parameters  $\varphi_n : [0, 1] \rightarrow SO(n)$ ,  $\varphi_m : [0, 1] \rightarrow SO(m)$  such that

- (1)  $\mathcal{L}(T) > 1 - c(n, m) \geq 1/2$ ,
- (2)  $\varphi_n|_T : T \rightarrow SO(n)$ ,  $\varphi_m|_T : T \rightarrow SO(m)$  are continuous,
- (3)  $|d_n(\varphi_n(s), \varphi_n(t)) - d_m(\varphi_m(s), \varphi_m(t))| < c(n, m)$  for any  $s, t \in T$ .

**Claim 3.8.** *There exist  $s_0, t_0 \in T$  such that  $d_n(\varphi_n(s_0), \varphi_n(t_0)) = \text{diam } SO(n)$ .*

*Proof.* Take  $A_0, B_0 \in SO(n)$  such that  $\text{diam } SO(n) = d_n(A_0, B_0)$  and define a map  $\psi : SO(n) \rightarrow SO(n)$  by  $\psi(A) := AA_0^{-1}B$ . Then,  $\psi_*(\mu_n) = \mu_n$  and  $d_n(A, \psi(A)) = \text{diam } SO(n)$  for any  $A \in SO(n)$ . Suppose that  $d_n(\varphi_n(s), \varphi_n(t)) < \text{diam } SO(n)$  for any  $s, t \in T$ . Then, we get  $\psi(\varphi_n(T)) \cap \varphi_n(T) = \emptyset$ , which leads to

$$\begin{aligned} \mu_n(\psi(\varphi_n(T)) \cap \varphi_n(T)) &= \mu_n(\psi(\varphi_n(T))) + \mu_n(\varphi_n(T)) \\ &= \mu_n(\psi^{-1}(\psi(\varphi_n(T)))) + \mu_n(\varphi_n(T)) \\ &\geq 2\mu_n(\varphi_n(T)) > 1. \end{aligned}$$

This is a contradicton and thus we complete the proof of the claim.  $\square$

By Claim 3.8, we obtain

$$\text{diam } SO(n) - \text{diam } SO(m) \leq |d_n(\varphi_n(s_0), \varphi_n(t_0)) - d_m(\varphi_m(s_0), \varphi_m(t_0))| < c(n, m),$$

which is a contradicton. This completes the proof of Lemma 3.7.  $\square$

*Proof of Proposition 1.2.* An easy caluculations show that  $2\sqrt{n-1} \leq \text{diam } SO(n) \leq 2\sqrt{n}$ . Therefore, supposing  $n_k \geq m_k$ , we have

$$\begin{aligned} \text{diam } SO(n_k) - \text{diam } SO(m_k) &\geq 2\sqrt{n_k-1} - 2\sqrt{m_k} \\ &= 2 \frac{n_k - m_k - 1}{\sqrt{n_k-1} + \sqrt{m_k}} \geq 2 \frac{C_3 - 1/\sqrt{k}}{\sqrt{C_1-1/k} + \sqrt{C_2}}. \end{aligned}$$

Thus, applying Lemma 3.7, we complete the proof.  $\square$

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